The Gerstenhaber Problem

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Motivation

Cayley-Hamilton Theorem

For $A \in \text{Mat}_{n\times n}(\mathbb{C})$ and $p_A(x) = \det(xI_n - A)$, we have $p_A(A) = 0$

Since $\deg(p_A(x)) = n$, then the $\mathbb{C}$-algebra generated by A has dimension at most n, with spanning set $\{I_n, A, \cdots, A^{n-1}\}$

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Notation

For pairwise commuting matrices $A_1, \cdots, A_k \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, we will denote the $\mathbb{C}$-algebra generated by them as $\mathbb{C}[A_1, \cdots, A_k]$

For $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ we have $\dim(\mathbb{C}[A]) \leq n$

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For $A, B \in Mat_{n\times n}(\mathbb{C})$ commuting matrices, can we get a dimension bound on $\mathbb{C}[A, B]$?

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Theorem (Gerstenhaber 1961)

If $A, B \in Mat_{n\times n}(\mathbb{C})$ commute, then $\dim(\mathbb{C}[A, B]) \leq n$

The Gerstenhaber Problem

Surely, this can't keep on going. What if we have three commuting matrices $A, B, C \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, is $\dim(\mathbb{C}[A, B, C]) \leq n$?

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The Gerstenhaber Problem

For pairwise commuting matrices $A_1, \cdots, A_k \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, when does the inequality $\dim(\mathbb{C}[A_1, \cdots, A_k) \leq n$ hold?

Bad things come in fours

When we look at $k = 4$ then the bound is known to fail.

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Example

Let e_i denote ith standard basis vector in $\mathbb{C}^4$, then the matrices given by $e_1 e_3^T, e_2 e_3^T, e_1 e_4^T, e_2 e_4^T$ generate a $\mathbb{C}$-algebra of dim $= 5$

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This construction can be extended for k commuting n by n matrices with $k, n \geq 4$, such that the algebra they generate is dim $> k$.

Algebraic reformulation

If we let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables, one can rephrase the Gerstenhaber Problem in terms of dimensions of S-modules.

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Proposition (Rajchgot, Satriano 2018)

The Gerstenhaber Problem holds if and only if for all S-modules N which are finite-dimensional over $\mathbb{C}$, we have

$$\dim S/\mathrm{Ann}(N) \leq \dim N \tag{1}$$

Algebraic reformulation

A common class of S-modules are monomial ideals; Indeed if $I \subseteq S$ is a monomial ideal, it is easily seen that (1) holds for $N = S/I$

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Definition

An n-dimensional *Young diagram* is a subset $\lambda \subseteq \mathbb{Z}_{\geq 0}$ such that for all $v \in \lambda$ and $w \in \mathbb{Z}_{\geq 0}$, if $w \leq v$ then $w \in \lambda$

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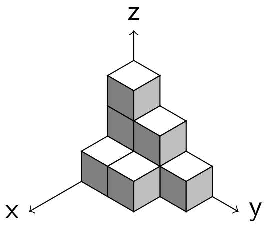
There is a inclusion reversing bijection between monomial ideals of $\mathbb{C}[x, y, z]$ and 3-dimensional Young diagrams.
For $I = \langle x^2, xy^2, y^3, xz, y^2z, yz^2, z^3 \rangle$, then S/I corresponds to



Result

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Theorem (Rajchgot, Satriano 2018)

If $I \subseteq S$ is an ideal of finite codimension, and N is an extension of S-modules satisfying the short exact sequence

$$0 \to S/I \to N \to S/\langle x, y, z\rangle \to 0$$

Then (1) holds

k=4: Revisited

One can realize the counterexample for $k = 4$ as the following module: Consider the ideals $I = \langle x_1, x_2 \rangle^2 + \langle x_3, x_4 \rangle$ and $J = \langle x_1, x_2 \rangle + \langle x_3, x_4 \rangle^2$ then we look at the module $S/I \oplus S/J$ obtained by gluing $(x_1, 0)$ to $(0, x_3)$, and $(x_2, 0)$ to $(0, x_4)$, that is

$$N = S/I \oplus S/J \; / \; \langle (x_1, 0) - (0, x_3), (x_2, 0) - (0, x_4) \rangle$$

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Where we can realize this pictorially, by looking at the following diagrams:

While identifying x_1 with x_3, and x_2 with x_4

General object of study

We can generalize this example, by considering monomial ideals $I \subseteq K$ and $J \subseteq L$ of S and an isomorphism $\phi : K/I \to L/J$ that sends monomials to monomials. Then define a S-module as

$$N = S/I \oplus S/J \;/\; \langle (f, -\phi(f)) : f \in K/I \rangle$$

An example

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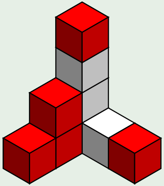
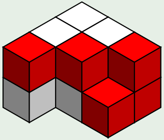
Example

Consider λ on the left and μ on the right, where we identify the red boxes between the two diagrams, which represent ν





Some combinatorics

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Recall $N = S/I \oplus S/J \ / \ \langle (f, -\phi(f)) : f \in K/I \rangle$.
It is straightforward to show that $\text{Ann}(N) = I \cap J$.
Which implies $\dim S/\text{Ann}(N) = \dim S/(I \cap J) = |\lambda \cup \mu|$.

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By counting, we get $\dim N = |\lambda| + |\mu| - |\nu|$.
So (1) is rephrased as

$$\begin{aligned} \dim S/\mathrm{Ann}(N) \leq \dim N &\iff |\lambda \cup \mu| \leq |\lambda| + |\mu| - |\nu| \\ &\iff |\nu| \leq |\lambda \cap \mu| \end{aligned}$$

A picture

We restate the Gerstenhaber Problem for our specific case as:

- Given two 3-dimensional Young diagrams λ and μ
- We identify a collection boxes between λ and μ, we name it ν
- We check if $|\nu| \leq |\lambda \cap \mu|$

A picture

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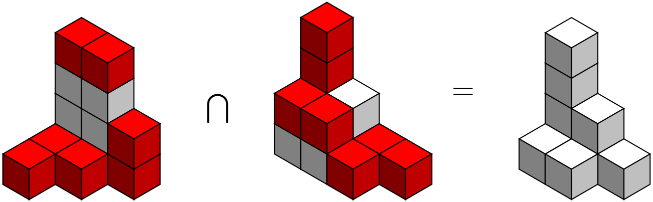

$$7 = |\nu| \leq |\lambda \cap \mu| = 9$$

What we showed

Definition

A *tower* is a subset of $\mathbb{Z}^3_{\geq 0}$ of the form $\{(x_0, y_0, z) : z_0 \leq z \leq z_1\}$ for some $x_0, y_0, z_0, z_1 \in \mathbb{Z}_{\geq 0}$

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For λ, μ, ν as before. If ν only consists of towers, then $|\nu| \leq |\lambda \cap \mu|$

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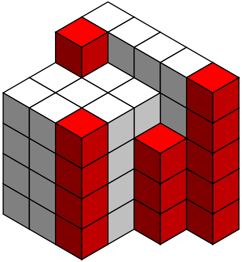
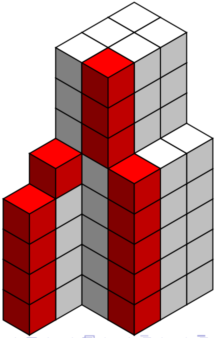




The proof

Given two modules defined by diagrams (λ, μ, ν) and (λ', μ', ν') we define the following order

Definition

We say $(\lambda, \mu, \nu) \leq (\lambda', \mu', \nu')$ if the following hold:

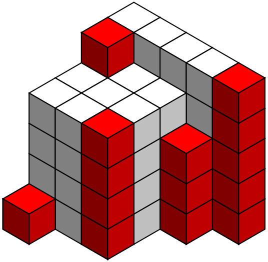
- $\lambda \subseteq \lambda'$ and $\mu \subseteq \mu'$
- there exists an injection $\iota : \nu \hookrightarrow \nu'$ that sends connected components to connected components
- $|\lambda \cap \mu| - |\nu| \leq |\lambda' \cap \mu'| - |\nu'|$

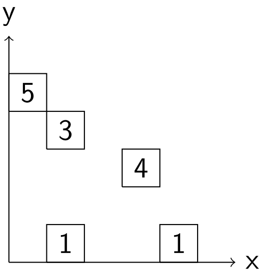
Reduction

This allows us to get rid of redundant information like so:

Floor plans

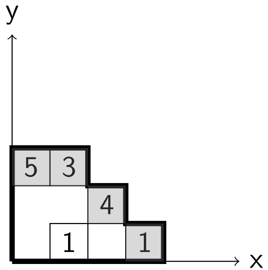
Then we project the diagrams onto the *xy*-plane, which we call *floor plans*:





Final step

We then look at minimal counterexamples with respect to the defined order, which allows us to restrict to the case where the *border* of floor plans consists of elements of ν:



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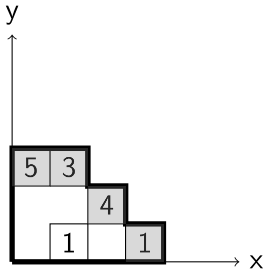

Using these restrictions, we manage to reduce a minimal counterexample to a smaller counterexample, contradicting minimality.

The End

Thank you!